The dynamics of bank runs: a simple cascade model*

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Abstract

This paper proposes a dynamic model in which bank runs arise as cascades of withdrawals. The aim is to better understand how bank runs emerge and develop dynamically, without imposing an exogenous sequence of actions. With bounded rationality, agents employ a switching strategy that combines strategic complementarity and heuristics. When a fraction of random agents withdraw, under the right conditions, some depositors withdraw preemptively in response, increasing the probability that other depositors will run subsequently. The model is able to characterize two distinct patterns of runs. Immediate runs develop instantly following the shock with a stable trajectory. On the contrary, sudden runs occur "out of nowhere", with massive withdrawals concentrate in a very short duration, after a period of apparent inactivity. We provide analytical calculation of the tipping point *i.e.* when the panic bursts out and determine the optimal time window for interventions.

Keywords: *bank runs; panic; cascades; tipping; bounded rationality*

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1 Introduction

Bank runs and panics are at the heart of financial crises. Gorton [2008] stressed that the global crisis started in 2007 was similar to a large-scale bank run. Allen and Gale [2009] compared the crisis to the "perfect storm" in which "seemingly unrelated events reinforced one another to produce an overwhelming cataclysm". To summarize the sentiment shared among economists: the crisis took everyone by surprise. The question that how large panics can suddenly emerge remains troubling. This paper aims to shed some light on this question.

Theoretical literature on panic-based bank runs, and panic events more generally, is built upon the coordination game framework of Diamond and Dybvig [1983]. The main driving force is strategic complementarity: in the panic equilibrium, all depositors withdraw because they expect others would also withdraw and the bank will fail.

While this elegant framework provides many insights to understand bank runs, there is one shortfall. Symmetric and simultaneous actions make it difficult to study the dynamics of bank runs. By design, the panic state is achieved instantly in equilibrium. However, sequentiality of actions is an important feature, as Brunnermeier [2001] pointed out that withdrawals are made sequentially in reality. Existing literature has paid little attention to some important questions: how withdrawals are made over time? How fast a bank defaults from a run? How depositors synchronize their actions? Better understanding of these matters might be useful to devise interventions in time of crisis.

This paper proposes a dynamic model of bank runs to address these issues. In a finite horizon, depositors can withdraw at any point in time. Depositors have private information on total withdrawal with some errors. As Bernanke [2013] observed, in time of crisis, agents often have to face Knightian uncertainty. Therefore, we assume that agents do not know the distribution of private information. Facing this large amount of uncertainty, agents make decisions by following a switching strategy that combines strategic complementarity and heuristics. A depositor withdraws when her perceived total withdrawal reaches a precautionary threshold, which is determined by the liquidity of the bank.

Bank runs in this model are purely panic-driven. When a fraction of random agents withdraw, under the right conditions, signals get worse and trigger preemptive withdrawals from some other depositors. The additional fraction of withdrawals in turn increases the probability to withdraw for remaining depositors. By this feedback mechanism, bank runs arise as dynamic cascades of sequential withdrawals.

The model generates two stylized patterns of bank runs. Immediate runs take place when withdrawals build up following a stable increasing pattern. On the contrary, after a period of apparent inactivity, sudden runs occur "out of nowhere" without any visible sign.

The second pattern of runs is interesting because it might explain the phenomenon commonly referred as "the calm before the storm". Sometimes, panics do not manifest immediately following a shock. Only tiny changes build up over time, then a generalized panic suddenly breaks out as if there is an unexpected shift at the aggregate level. The idea can be illustrated with a popular game called Jenga. A wooden tower is constructed from removable rectangular blocks beforehand, then each player takes turn to remove one block at a time, until the tower collapse. In the early stage, when each block is removed, the fragility of the tower increase by an imperceptible margin. From a moderate distance, it would be impossible to tell whether the tower has been altered. At some critical point, where enough blocks are removed, the tower becomes visibly unstable. Removing one more block would make the tower collapse.

Our paper contributes to the literature in two directions. First, the model is able to replicate and characterize the patterns of bank runs that can be observed. It provides a possible explanation on why massive withdrawals suddenly occur, as if depositors synchronize their actions. An important result of the paper is the explicit computation of the tipping point, where the panic bursts out. To our limited knowledge, this issue has not been addressed in existing theoretical models. Second, this paper offers a novel approach to study bank runs and panics more generally. Bank runs arise as path-dependent cascades rather than mis-coordination. The model is simple and serves as an attempt toward building a quantitative model that might be able to detect panic-sensitive patterns.

With respect to existing literature, this paper is linked to both empirical and theoretical researches on bank run. Recent empirical works showed that the sequentiality of withdrawals is important and has influences on the outcomes (Schotter and Yorulmazer [2009]). Furthermore, there are evidences that decision of depositors are affected by the past withdrawals (Garratt and Keister [2009], Kiss et al. [2012]). Among a few exceptions, Gu [2011] proposed a bank run model with sequential actions. However, in her model, only one agent can take action at a time. The majority of existing theoretical models are built upon simultaneous coordination game and do not take into account these features. Our model introduces continuous, unconstrained sequence of actions and partial observability of past withdrawals.

Although conceptually different, the model presented here share some features with theoretical work on global games (see Carlsson and Van Damme [1993], Morris and Shin [2001]). The framework of global game has been applied to model panic events such as panic-based bank run (Goldstein and Pauzner [2005]) and currency attack (Morris and Shin [1998]). These models use a setting of coordination game with structural uncertainty, where payoffs are random variables and agents receive signals on the payoffs. The common mechanism with our model is the switching strategy: agents choose an action by default and switch to the other action if they observe a signal above a critical threshold. However, the central assumption of these models is that the structure of information is common knowledge, such that agents can infer the distribution of signals of others to apply iterative deletion of dominated strategies. In our model, agents have bounded rationality and do not know the complete structure of information. The threshold comes from individual perception of the aggregate state of the economy, rather than strategic deduction of beliefs. Our assumption retains rational behaviors while simplifies the strategic aspect of the problem, to focus more on the dynamic aspect of panics.

Also, this paper is inspired by models of collective behavior (Granovetter [1978]) and dynamic diffusion (Bass [1969]). The pioneer work of Granovetter [1978] introduced a global interaction "all-to-all" mechanism: agents use simple rules to adopt an action based on the "popularity" of that action on the aggregate scale. Each individual action can affect the decision of all other agents. Under the right conditions, the same action is adopted at the individual level over time. The collective behavior then emerges as if there is a shift on the aggregate scale. In a different setting, Bass [1969] modeled the speed of diffusion of a new product when a fraction of agents try to imitate others. The diffusion process is path dependence: new adoptions depend on the fraction of past adoptions made by others. Our model makes use of these features to model the dynamics of bank runs as a cascade mechanism.

The remainder of the paper is organized as follows. Section 2 presents the model, then establishes the dynamics of bank runs and characterizes these dynamics. Section 3 discusses some real-world examples and policy implications. Section 4 concludes the paper.

2 The Model

2.1 Setting

The economy. A continuum of agents are depositors to a common bank. The time horizon T is finite with n periods. The time step is denoted by $\Delta t = \frac{T}{n}$. If $\Delta t = 1$, then time is indexed by $t \in \{1, 2, \ldots, n\}$ and $n = T$.

Depositors. By default, each agent has one unit of deposit in the bank. Depositors decide to keep their deposit in the bank (wait) or to withdraw (run) in each period. To simplify the analysis, each depositor can only withdraw all of her deposit at once. Depositors who withdrew cannot put their deposit back in the bank and become inactive. Let $r_t \in [0, 1]$ denote the fraction of agents who withdraw at date t and $R_t = \sum_{k=0}^{t} r_k$ with $R_t \in [0, 1]$ denote the total fraction of withdrawals up to the end of date t.

In each period, depositors have private information on the total fraction of withdrawal : $\tilde{R}_{it} = R_{t-1} + \tilde{z}_{it}$. The distribution of private information represents diversity in opinion and capacity to process aggregate information. For technical simplicity, $\tilde{z}_{it}$ are i.d.d and uniformly distributed in the interval $[-\varepsilon, \varepsilon]$. In the remaining of the paper, $\tilde{R}_{it}$ are labeled as signals and $\tilde{z}_{it}$ are labeled as noises.

Bank & deposit contract. A fraction of deposits $L \in (0, 1)$ is kept as liquidity reserve to meet liquidity demand at short term. The remaining fraction of deposits is invested in illiquid assets. The investment yields positive profit with certainty at T. The bank has better investment opportunity and offers demandable debt-deposit contracts to depositors. Deposit contracts have maturity T. The time horizon T represents the maturity mismatch between short-term demandable deposits and the long-term investment.

The model assumes an implicit deposit contract and agents behave as if they are offered the contract described here. For each unit of deposits, there is a positive return at maturity, the payoff is $C > 1$ at $t = T$. At any interim period, depositors can choose to forgo the interest at maturity to get back their deposit, the short-term payoff is normalized to 1. Thus, L equals the fraction of the population to which the bank can pay back at short term without going bankrupt. If the total withdrawal is greater than the available liquidity at any period $t^* < T$, the bank defaults and liquidates the long-term investment. Depositors are paid in a first come first served basis. Each late runner get a payoff $c < 1$ until the liquidation proceeds are depleted. If $0 < c < 1 < R$, the structure of the payoffs is sufficient to generate strategic complementarity. The payoffs *per se* have little influence on the results of the model.

Timeline. At $t = 0$, a fraction r_0 of random agents withdraws. At $t > 0$, active agents choose to wait or withdraw in each period. The bank fails if at any time $t^* < T$, total fraction of withdrawals exceeds liquidity reserve *i.e.* $R_t > L$. Otherwise, the bank survives. The parameter r_0 reflects a "random" liquidity shock, when the bank already committed to the investment.

Decision-making. Since the distribution of noises is not assumed to be common knowledge, each particular depositor can not infer the distribution of signals of other agents using her own signal. Payoff maximization will depend on individual-specific additional priors. The solution of the dynamic optimization problem would be subjected to a large confidence interval and require tremendous amount of computational power.

Subjected to this large amount of uncertainty, agents follow a switching strategy to approximate payoff maximization. Let a_{it} the action of agent i in period t. If $a_{i,t-k} \neq$ withdraw, $\forall k = 1, 2, \ldots, t$ then:

$$a_{it}(\tilde{R}_{it}, \tau_i) = \begin{cases} \text{withdraw} & \text{if } \tilde{R}_{it} > \tau_i \\ \text{wait} & \text{otherwise} \end{cases}$$

where $\tilde{R}_{it}$ is taken as the perceived expected total withdrawal and τ_i is the precautionary threshold.

In a broad sense, the threshold reflects individual perception on the fragility of the bank, as if agents "discount" the true liquidity level. Let θ_i be the discounting factor of agent i. The precautionary threshold is obtained by a convex monotonic transformation $f_{\theta_i} : L \to \tau_i$ such that 3 conditions must be satisfied:

1. $\tau_i < L$
2. if $L \to 0$, then $\tau_i \to 0$
3. if $L \to 1$, then $\tau_i \to 1$

These 3 conditions make sure that individual behaviors are rational: (1) agents always withdraw before the perceived total withdrawal reaches the true liquidity reserve; (2) when the bank has little liquidity, agent has low incentive to wait, as any small fraction of withdrawals could make the bank fail; (3) vice-versa, agents have high incentive to wait when the bank has high liquidity reserve.

Specifically, the precautionary threshold is given by:

$$\tau_i = L^{\theta_i}$$

with $L \in (0, 1)$ and $\theta > 1$.

The discounting factor reflects characteristics of depositors that influence their willingness to wait. To simplify mathematical operations, let assume that depositors share a unique discounting factor, such that $\theta_i = \theta$. This unique value can be regarded as the mean of the distribution of individual values. Therefore, the switching threshold has a unique value:

$$\tau_i = \tau = L^\theta$$

A higher value of θ implies a larger precautionary gap, such that agents are more sensitive to withdraw. Figure 1 illustrates an example of precautionary threshold. Empirically, the parameter θ can be linked to bank-client relationship, for example. Iyer and Puri [2012] found that depositors with better relationship with the bank delay their decisions to withdraw in a bank run.

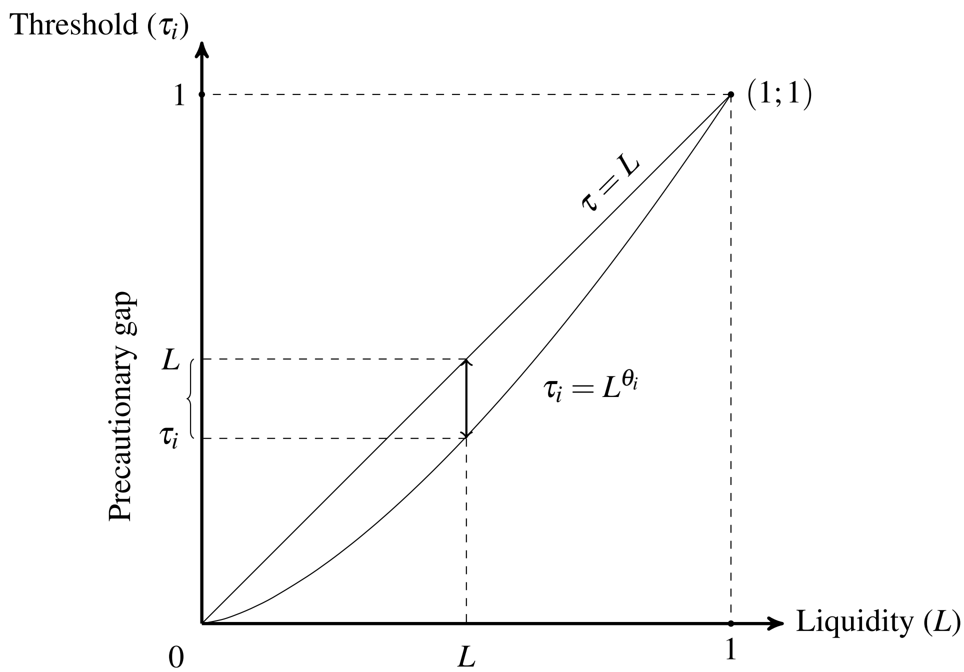

Figure 1: Precautionary threshold, $\theta > 1$. Higher value of θ make the curve bend downward, resulting in a larger precautionary gap.

The decision-making process is built upon both strategic complementarity and heuristics. Strategic complementarity reflects rationality in individual decisions. As withdrawals drain liquidity, the bank becomes more sensitive to failure. Depositors withdraw preemptively as if their expected payoffs are decreasing with the perceived total withdrawal. However, given the large amount of uncertainty, the threshold can not be strategically determined. This model assumes that agents have bounded rationality and thresholds are determined by characteristics of depositors, as suggested by empirical evidences. This assumption does not rule out the case that thresholds can have a common value, which coincides with the unique threshold determined by strategic deduction of beliefs as in global games (interested readers can see Morris and Shin [2001] for more details).

Intuitively, the switching strategy can be understood as a set of simple rules. First, agent avoid running too late, when the bank already failed. Secondly, agents also avoid running too early, when liquidity reserve is sufficiently high compared to the perceived total withdrawal. Running early is to deny the highest payoff when the chance of bankruptcy in the next period is low. Therefore, agents wait as long as possible, in an attempt to get the highest payoff. Only when the perceived

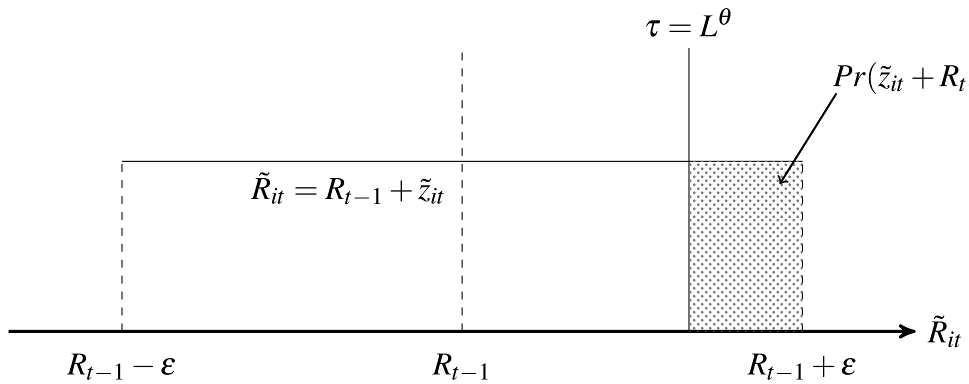

Figure 2: Probability to withdraw. When total withdrawal increases, the distribution shifts to the right, making the probability larger.

total withdrawal becomes large enough, agents will run to avoid losses. There are experimental evidences that depositors use cutoff thresholds to make withdrawals (Garratt and Keister [2009]).

Interactions. In this model, the dynamics are driven by a feedback mechanism. The aggregate information R_t acts a global interaction device. Whenever a fraction of depositors with high signals withdraw early, these withdrawals make R_t slightly larger, thus increases the chance to have a larger signal $\tilde{R}_{i,t+1}$ to all other agents. This stochastic feedback mechanism can generate a cascade of actions, when a relatively small fraction of withdrawals has positive probability to induce a larger fraction of depositors to run. Over time, depositors might be caught up in a generalized panic and forced to run.

2.2 Dynamics of withdrawals

Given the switching strategy, the probability that an agent withdraws in period t is $Pr\left(\tilde{R}_{it} > \tau\right)$. Since $\tilde{R}_{it} = R_{t-1} + \tilde{z}_{it}$, signals are uniformly distributed in $[R_{t-1} - \varepsilon, R_{t-1} + \varepsilon]$. Figure 2 illustrates the individual probability to withdraw.

As a thought experiment, the analysis begins with an extreme case, in which $\varepsilon = 0$. The distribution of signals collapses into a vertical line. When the noises disappear, agents have the same information and they always know the true value of total withdrawal. Moreover, agents also have the same threshold. If the shock is larger than the unique threshold, then everybody withdraws. Otherwise, nobody withdraws at all. The outcomes are two symmetric equilibria: all agents run if $r_0 > \tau$ and all agents wait if $r_0 < \tau$. It is important to notice that the economy reaches equilibrium state immediately in this case, there is no sequential withdrawals.

In what follows, we assume that $\varepsilon > 0$. By the law of large numbers, the fraction of withdrawal in period t is the individual probability to withdraw times the fraction of remaining (waiting) agents :

$$r_t = Pr\left(\tilde{z}_{it} + R_{t-1} > \tau\right)\left(1 - R_{t-1}\right) \tag{1}$$

Given that signals are uniformly distributed, the probability to withdraw is zero if τ is greater than the highest possible signal $R_{t-1}+\varepsilon$ at any period t. Hence, Lemma 1 characterizes the no-withdrawal condition.

Lemma 1. *Given an initial shock r_0, no additional withdrawal is made if the following condition holds:*

$$L^{\theta} \geq r_0 + \varepsilon$$

Proof. From equation (1): $Pr(\tilde{z}_{i1} + r_0 > \tau) = 0$ when $r_0 + \varepsilon \leq L^{\theta}$, then $r_1 = 0$ and $R_1 = r_0$. By forward induction: $\forall t > 0, r_t = 0$ and $R_t = r_0$. □

Lemma 1 states that if the discounted liquidity reserve is higher than the maximum signal, no depositor will withdraw at all. In other words, following the initial withdrawals, if the most pessimistic depositor (with the highest signal) does not think that the bank will fail in the next period, then no one withdraws. Given that no depositor withdraws in the past period, the same argument applies recursively and no withdrawal is ever made.

Otherwise, given the condition $L^{\theta} < r_0 + \varepsilon$, there is always a positive fraction of withdrawal every period. The dynamics of sequential withdrawals are described by the following system:

$$\begin{cases} r_{t+1} &= \frac{1}{2\varepsilon}(\varepsilon - L^{\theta} + R_t)(1 - R_t) \\ R_t &= \sum_{j=0}^{t} r_j \end{cases} \tag{2}$$

with initial condition $R(0) = r_0$ and $1 > r_0 > L^{\theta} - \varepsilon$.

The system (2) is similar to a discrete logistic map. The sequence $(r_0, r_1, \ldots, r_t)$ may exhibit chaotic behaviors and has no closed-form solution. To simplify the analysis, an approximation in continuous time is used for the remaining of the paper.

For any time step $\Delta t < 1$, the fraction of withdrawal *per time step* is given by

$$r_{t+\Delta t} = \frac{1}{2\varepsilon}(\varepsilon - L^{\theta} + R_t)(1 - R_t)$$

By definition, $r_{t+\Delta t}$ is the change in the cumulative fraction of withdrawal: $r_{t+\Delta t} = R_{t+\Delta t} - R_t$. The change in cumulative fraction of withdrawal *per time step* is given by

$$\frac{R_{t+\Delta t} - R_t}{\Delta t} = r_t$$

For a finite time horizon T, time steps become smaller when the number of periods increases. Small time periods allow for an approximation of the system (2) in continuous time.

$$\underset{\Delta t \to 0}{Lim}\left(\frac{R_{t+\Delta t} - R_t}{\Delta t}\right) = \frac{dR}{dt} = r_t$$

Therefore, the accumulation of sequential withdrawals is given by the following differential equation (law of motion):

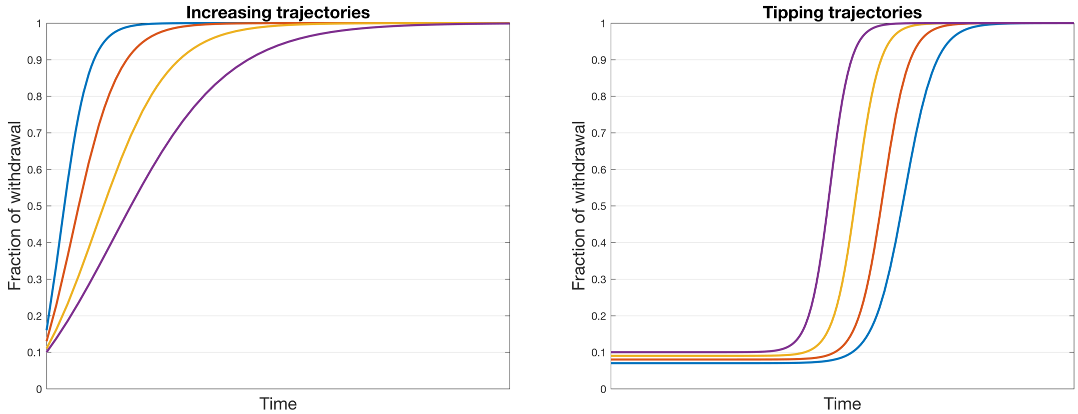

Figure 3: Stylized dynamics of bank runs.

$$R'(t) = \frac{1}{2\varepsilon}(\varepsilon - L^{\theta} + R(t))(1 - R(t)) \tag{3}$$

with conditions $1 > r_0 > L^{\theta} - \varepsilon$.

As the RHS of equation (3) is a quadratic concave function of $R(t)$ which becomes zero at $L^{\theta} - \varepsilon$ and 1, $R(t)$ is always non-decreasing. The solution of equation (3) will provide a complete description of the dynamics of sequential withdrawals.

Proposition 1. *(Dynamics of withdrawals) The total withdrawal at time t is given by*

$$R(t) = \frac{\left(r_0 + \varepsilon - L^{\theta}\right) e^{ht} - \left(\varepsilon - L^{\theta}\right)(1 - r_0)}{\left(r_0 + \varepsilon - L^{\theta}\right) e^{ht} + 1 - r_0} \tag{4}$$

where $h = \frac{1}{2\varepsilon}(\varepsilon - L^{\theta} + 1)$*, with initial condition* $r_0 > L^{\theta} - \varepsilon$.

Proposition (1) states that for a range of parameters, the cumulative total fraction of withdrawals can be described by a generalized logistic function. From equation (4), when the no-withdrawal condition does not hold, the economy is set into motion toward the steady-state $R(t) = 1$ by a cascade mechanism: any positive fraction of withdrawals will induce more withdrawals.

Most importantly, Proposition 1 shows how withdrawals are made over time. There are two stylized dynamics of runs depicted in Figure 3. For some cases, runs are apparent immediately after the shock and follow a stable increasing trajectory. On the contrary, after a period of inactivity, sudden runs occur "out of nowhere" without any visible sign. At first, the shock may not seem to trigger any visible fraction of withdrawals, then a massive withdrawal takes place. In what follows, we refer to sudden run as "tipping".

Immediate runs occur when the shock is relatively high compared to liquidity reserve. On the contrary, sudden runs require a certain degree of balance between liquidity reserve and shock. Tipping only occur when parameters break the no-withdrawal condition by a small margin. Intuitively, when $r_0 = \varepsilon - L^{\theta}$, there is no withdrawal and $R(t)$ is a flat line. With a small perturbation, infinitesimal fractions of withdrawals take place, each time raising the probability to withdraw by a small margin. As the term $(1-R(t))$ remains close to 1, the cumulative process builds up with increasing speed. When the panic becomes visible, its growth rate is already high. This high growth rate produces an apparent jump in total withdrawal, compared to the previous periods. This is the "surprise effect": following imperceptible changes, the cascade bursts out in a very short time window, making a bank run seemingly occurs out of nowhere.

2.3 Defaulting patterns

Given the evolution of cumulative withdrawal and the liquidity reserve, it is possible to determine the hitting time t^* when the liquidity reserve is completely exhausted. Visually, it is the moment that the curve $R(t)$hits the horizontal line L. Proposition 2 gives existence conditions and characterizes the hitting time.

Proposition 2. *(Hitting time) If the conditions $r_0 < L < (r_0+\varepsilon)^{\frac{1}{\theta}}$ are satisfied, the hitting time is well defined and given by*

$$t^* = \frac{2\varepsilon}{\varepsilon - L^\theta + 1} \ln \left[\frac{\left(\varepsilon - L^\theta + L\right)(1-r_0)}{\left(r_0 + \varepsilon - L^\theta\right)(1-L)} \right] \tag{5}$$

In this model, the bank only defaults within the time horizon T. Therefore, the hitting time is also the defaulting time if $0 < t^* < T$.

Following equation (5), the condition $L > r_0$ ensures that t^* will only take strictly positive values. The lower bound of the defaulting time is 0, when $L = r_0$. By design, this condition is consistent with the definition of default: if the initial shock already depletes the liquidity reserve, the bank fails immediately. There is no upper bound for t^*. If L approaches the limit $(r_0+\varepsilon)^{\frac{1}{\theta}}$, the hitting time tends to infinity. This result agrees with Lemma 1: when liquidity is too high, no additional withdrawal is made, thus the bank never defaults. Proposition 2 establishes that if the conditions $r_0 < L < (r_0+\varepsilon)^{\frac{1}{\theta}}$ holds, continuous withdrawals are made following the dynamics described in Proposition 1 and the exact time that total withdrawal reaches the liquidity level is explicitly given. The following corollaries analyze the survival duration of the bank with respect to the parameters, when the conditions in Proposition 2 hold.

Corollary 1. *The hitting time t^* is strictly increasing with L and strictly decreasing with r_0.*

If a bank run occurs, more liquidity helps to satisfy more withdrawals, thus increasing the time needed to deplete liquidity reserve. On the contrary, higher shocks imply higher initial conditions for the dynamic process $R(t)$, making cumulative withdrawal reach any predetermined level sooner with the same speed.

Corollary 2. *The hitting time t^* is decreasing with ε, for small values of ε such that $t^*(\varepsilon) < \lambda(\varepsilon)$, with $\lambda(\varepsilon) = \frac{2\varepsilon^2(L-r_0)}{\left(r_0+\varepsilon-L^\theta\right)\left(L+\varepsilon-L^\theta\right)\left(1-L^\theta\right)}$.*

The dependency of t^* with respect to ε is more ambiguous, technical details are provided in the Appendix. Intuitively, this result follows the mechanism depicted in Figure 2. When ε is small initially, an increase in ε significantly boost the probability to withdraw in the early stage, leading to a higher early growth rate of cumulative withdrawal. Everything being equal, liquidity reserve is depleted sooner when withdrawals accumulate faster.

The next step is to find out how much liquidity the bank must hold in order to survive the time horizon T. From Proposition 2, it is possible to determine the liquidity level that will be depleted at a given time. However, solving the equation (5) for L involves a Lambert W function, which has no analytical solution. Thus, it is necessary to use an implicit function.

Definition 1. Define the set I such that for any $L \in I$ then $r_0 < L < (r_0 + \varepsilon)^{\frac{1}{\theta}}$. Let the function of hitting time $\varphi : I \to \mathbb{R}_+$ be defined by the equation (5): $t^* = \varphi(L)$. Because $\varphi(L)$ is strictly increasing in L, the inverse function of hitting time is uniquely defined by $\varphi^{-1} : \mathbb{R}_+ \to I$

$$L = \varphi^{-1}(t^*)$$

In other words, for a specific value t^*, the function $\varphi^{-1}(t^*)$ gives the value of L that satisfies the equation (5) such that $R(t^*) = L$. In what follows, we use the notation $L^*(t)$ to indicate the liquidity level that will be depleted at time t.

Because $\varphi(L)$ is strictly increasing in L, thus $L^*(t)$ is also strictly increasing in t. In other words, to survive longer in a bank run, the bank must hold more liquidity. Using this monotonicity of inverse function, it is possible to characterize the defaulting patterns and occurrence of bank runs.

Proposition 3. *(Defaulting patterns) For a given triplet (L, r_0, ε), with any values of t and T that satisfy $0 < t < T$, the following conditions hold:*

$$r_0 < L^*(t) < L^*(T) < (r_0 + \varepsilon)^{\frac{1}{\theta}}$$

such that a bank run occurs when $r_0 < L < (r_0 + \varepsilon)^{\frac{1}{\theta}}$ and the bank always defaults from a bank run when $r_0 < L < L^*(T)$.

The proof is straightforward and therefore omitted. Proposition 3 establishes four possible outcomes for the bank when hit by a “random” liquidity shock. The results are depicted in Figure 4. When liquidity reserve is lower than or equal to the shock, the bank defaults immediately. By contrast, when liquidity is above the no-withdrawal threshold, the bank never defaults and the economy reaches a steady state immediately. When liquidity is moderate compared to the shock, a bank run occurs dynamically. The bank fails if the liquidity reserve is lower than a critical value $L^*(T)$, because withdrawals will deplete the liquidity before the maturity. The relative distances between the critical values are important because they determine the probability of bank run and the probability of default.

2.4 Comparative statics

The next step is to study how the regions in Figure 4 vary with the main parameters. Even without an explicit expression of $L^*(t)$, it is possible to derive properties of this function to advance the analysis.

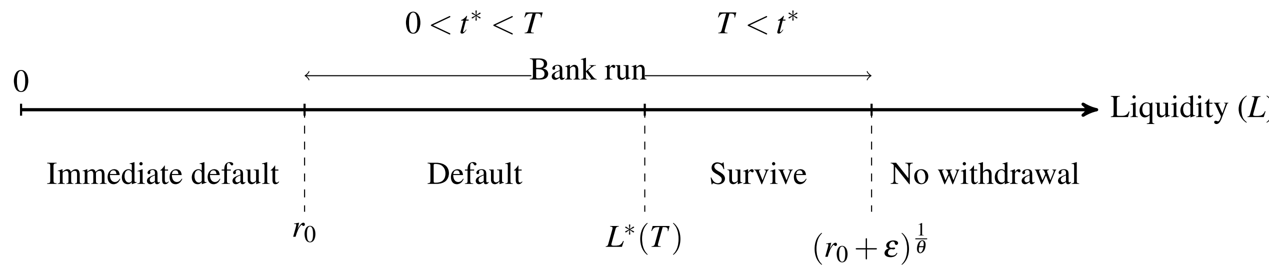

Figure 4: Defaulting patterns.

Corollary 3. *Longer maturities (T) strictly increase the probability of default from a bank run*

The proof is straightforward: since the function $\varphi^{-1}(\cdot)$ is strictly increasing, for any $T' > T$, it is true that $L^*(T') > L^*(T)$. While the conditions for run do not change, when T increases, we have $L^*(T) \to (r_0+\varepsilon)^{\frac{1}{\theta}}$ such that the survival region shrinks. When the maturity increases, if a run is triggered, it has more time to reach the predetermined liquidity level. If T is large enough, the chance to survive a run is almost zero. Recall the assumption that the bank cannot obtain additional liquidity in the time horizon T. In practice, banks can sell liquid assets or borrow at short term to face withdrawals, but these measures are costly and may deteriorate fundamentals of the bank, increasing the speed of withdrawals. Thus, reducing the illiquid time horizon could be an effective measure against panic runs. This result may explain why financial institutions without access to low-cost liquidity (such as Fed funds) increasingly turn to short-term funding. However, relying too much on short-term funding can lead to inefficient outcomes, such as the "maturity rat race" pointed out by Brunnermeier and Oehmke [2013].

Corollary 4. *Larger shocks (r_0) strictly increase the probability of default. However, the probability of bank run increases with r_0 if and only if $r_0 < \theta^{\frac{\theta}{1-\theta}} - \varepsilon$.*

Using the implicit function theorem, it is possible to show that $\frac{\partial L^*}{\partial r_0} > 0$, therefore the default region expands with higher r_0. However, increasing r_0 will also shift the other critical values forward. When the condition $r_0 < \theta^{\frac{\theta}{1-\theta}} - \varepsilon$ holds, we have $\frac{\partial}{\partial r_0}(r_0+\varepsilon)^{\frac{1}{\theta}} > 1$ such that the bank-run region expands with higher r_0. Otherwise, the critical value $(r_0+\varepsilon)^{\frac{1}{\theta}}$ advances slower than the first boundary, making the bank-run region shrink.

The probability of bank run increases simply because the no-withdrawal condition is tightening: for larger shocks, it requires more liquidity to keep even the most pessimistic depositor from withdrawing. Everything being equal, a larger shock provides a higher initial condition for the dynamic process $R(t)$, as if the trajectory of $R(t)$ shifts to the left with a higher starting point. For small shocks, this mechanism will increase the probability of default because total withdrawal can reach a higher level of liquidity for the same time horizon. However, above a certain threshold, large shocks will more likely make the bank default immediately rather than trigger a bank run.

Corollary 5. *Larger magnitudes of noises* (ε) *increase both the probability of default and the probability of bank run, for small values of* ε.

Similar to the precedent result, it is trivial that $\frac{\partial}{\partial r_0}(r_0+\varepsilon)^{\frac{1}{\theta}} > 0$. The probability of bank run increases with ε because depositors have more "extreme" signals, such that withdrawals can be triggered even when liquidity is high. For small values of ε, recall that $t^*(\varepsilon)$ is a decreasing function, then it is possible to show that $\frac{\partial L^*}{\partial \varepsilon} > 0$. Intuitively, higher dispersion of private signals increases the chance to draw high signals, especially in the early stage, thus making the probability to withdraw larger. This early boost of withdrawals makes the bank more likely default.

2.5 Abruptness & tipping point

The previous results provide some insights on the factors that facilitate the occurrence of bank runs. Given that the bank-run conditions hold, one interesting question is how the run would occur. Figure 3 showed that there are two patterns: immediate run and sudden tipping. This section studies the abruptness of these dynamics.

The first element is the steepness of the fast ascending phase of the trajectory. A steeper curve implies that the run is more abrupt: a large fraction of withdrawals is concentrated in a small time window. Given that cumulative withdrawal follows a generalized logistic function described by equation (4) , the term e^{ht} mainly determines the speed of growth of the trajectory $R(t)$. This leads to the following proposition.

Proposition 4. *The abruptness of runs is decreasing with liquidity* (L) *and magnitude of noises* (ε)*, given that the bank-run conditions hold.*

Precisely, the higher value of the exponent $h = \frac{(\varepsilon - L^{\theta}+1)}{2\varepsilon}$, the trajectory will be steeper in its fast ascending phase. It is straightforward that h is decreasing with both L and ε. The first element is obvious: a higher liquidity level makes a higher threshold, such that depositors withdraw less in every time step. Therefore, the cumulative withdrawal takes more time to build up.

The second result may seem counter-intuitive. The explanation resides in the law of motion, equation (3) . Given that the bank-run conditions hold, a lower magnitude of noises decreases the probability to withdraw in the early stage. This effect is more apparent in the beginning, when the probability to withdraw is minimal. Therefore, the early growth rate of the trajectory is significantly slower. This in turn makes the probability to withdraw increase very slowly. However, slow early growth preserves the fraction of waiting depositors. When cumulative withdrawal is large enough to make an apparent shift in the probability to withdraw, the fraction of waiting depositors is still large. Therefore, the multiplicative effect of these factors produces a large fraction of withdrawals in a small time window. On the contrary, large magnitude of noises generates high early growth rate. When the probability to withdraw reaches a high level, the fraction of waiting depositors is already small. The speed of growth is more stable because the two terms in the law of motion balance each other over time.

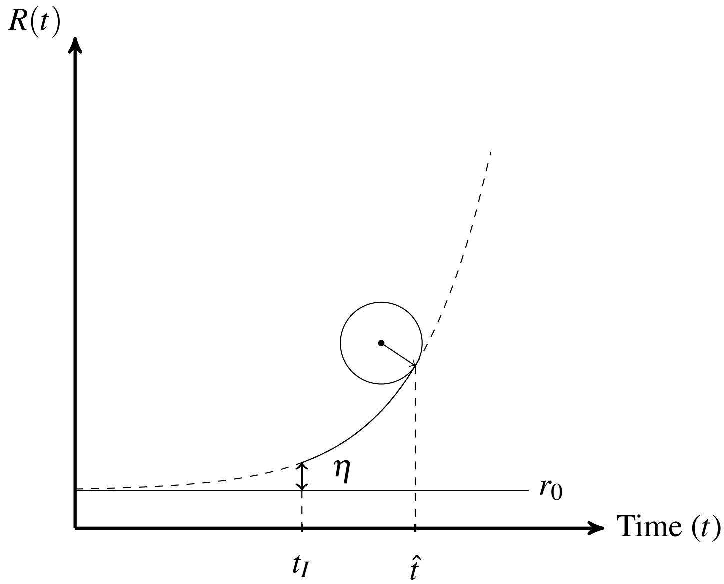

Figure 5: Irruption time & tipping point. The graphic illustrates a magnified segment of a tipping trajectory.

Given that bank runs can be abrupt, there is little to do for an immediate run. However, sudden runs are interesting for two reasons. First, these sudden runs are triggered by small shocks, they are hard to detect and could be ignored until it is already too late. Second, as the early growth rate of the panic is very low, it requires less costly interventions to avoid bankruptcy, if interventions are made on time. Further analysis of tipping trajectories requires some definitions, illustrated in figure 5.

Definition 2. Let $\eta > 0$ be an arbitrarily small number. Define the irruption time (t_I) as the moment in which the trajectory $R(t)$ leaves the neighborhood $(r_0 + \eta)$. Thus, t_I is the solution of the equation $R(t_I) - r_0 = \eta$.

Proposition 5. *If $r_0 < \frac{1-\varepsilon+L^\theta}{2}$, the irruption time is given by*

$$t_I(\eta) = \frac{2\eta\varepsilon}{(1-r_0)\left(\varepsilon - L^\theta + r_0\right)} \tag{6}$$

Intuitively, t_I is the moment in which a potential panic becomes noticeable, by a predefined critical neighborhood. It possible to derive the theoretical upper bound for t_I in what follows.

Definition 3. By differential geometry, define the *curvature* κ of the graph $R(t)$ at a specific point $(t, R(t))$ as the sensitivity of the slope of the tangent lines around that point[1]. Specifically, the

[1] interested readers can see Kline [1998] for more details

curvature of the graph $R(t)$ is given by

$$\kappa(t)=\frac{R''(t)}{\left(1+(R'(t))^2\right)^{\frac{3}{2}}}$$

Define the *tipping point* $(\hat{t})$ as the moment where the curvature of the graph $R(t)$ is maximum.

Intuitively, suppose that we approximate each equal segment of the graph $R(t)$ with a tangent circle. Higher curvature at a point implies that the tangent circle is smaller, which makes the tangent lines "turn" faster around that point. Therefore, the change in speed is maximal where the trajectory $R(t)$ bends the most. Highest curvature implies that the acceleration of the crisis is maximal at that point.

Proposition 6. *The tipping point is given by*

$$\widehat{t}=\frac{2\varepsilon}{m+1}\ln\left[\frac{1-r_0}{r_0+m}\left(\frac{\frac{1+m}{2}-\sqrt{r}}{\frac{1+m}{2}+\sqrt{r}}\right)\right]$$

with $m=\varepsilon-L^{\theta}$ *and* r *is the smallest non-negative root of* $N(r)=0$, *where*

$$N(r)=3r^3-5\left(\frac{m+1}{2}\right)^2r^2+\left(\left(\frac{m+1}{2}\right)^4-\frac{3}{b}\right)r+\left(\frac{m+1}{2}\right)^6+(m+1)^2\varepsilon^2$$

Detailed calculation are given in the Appendix. Theoretically, it is possible that the tipping point is close to zero or even negative. This simply indicates that under some conditions, such as a noticeably large shock, the acceleration of the panic is already very high at the beginning. Otherwise, the tipping point is positively defined.

The tipping point $(\hat{t})$ marks the end of the early stage of the panic, beyond which the panic will burst out massively. The trajectory $R(t)$ is most sensitive at this point. Any small perturbation near the tipping point could substantially change the course of the trajectory. On the contrary, the iruption time (t_I) marks the perceptible beginning of the panic. The gap between t_I and $\hat{t}$ represents the optimal time window for interventions.

3 Discussions

From an empirical perspective, we will discuss some real-world examples to illustrate the two patterns of runs shown in the model.

The first pattern is immediate run, where a panic takes off instantly after the shock. One example is the Russian-LTCM crisis in 1998. LTCM was one of the largest investment firms of Wall Street. Its success was spectacular by the end of 1997. However, in August 1998, their fortunes changed when Russia unexpectedly defaulted on its government debts and devalued its currency. Being highly leveraged, LTCM lost half of its value in one month. Despite the small scale of the loss in total value, the russian default triggered an instant panic in financial markets. The cascade pattern

was recognizable: investors "run" away from risky assets, make asset prices depreciate and induce more runs (sales). If LTCM was liquidated, the resulted drop in asset prices could make many other institutions default. The Federal Reserve Bank of New York organized a bailout to avoid the potential systemic crisis.
The second pattern is tipping, where a run suddenly occurs after a period of apparent calm following the shock. One example is the run on Lehman Brothers in repurchase agreement (repo) markets in September 2008 (Copeland et al. [2014]). Three months prior to the bankruptcy, Lehman Brothers reported unprecedented losses. However, there was no panic. The repo division of Lehman Brothers was able to secure uninterrupted funding by tri-party repos with unchanged borrowing conditions for weeks. Repo remained as one of the main sources of funding for Lehman Brothers. Then it suddenly collapsed : in 5 days, investors pulled out massively, took away about 40% of short-term funding. This synchronized withdrawal pushed Lehman Brothers to declare bankruptcy subsequently.

From a policy perspective, the model showed that there is an optimal time window for interventions. If one admits the cascade argument, then panics are path dependent. This implies that it is possible to dissolve or at least dampen some large crises with relatively small effort, if interventions are made at the right moment. In this model, we have identified the tipping point, above which the panic will burst out. It would be either too late or very costly to react if this point is reached. It is obvious that this simple model cannot provide the exact mathematical description of the underlying mechanism, it could only generate the stylized dynamics. However, it puts forward the importance of building a descriptive model that able to identify panic-sensitive patterns to minimize the required efforts when interventions are inevitable.

4 Conclusion

This paper has studied the dynamics of bank runs in a model that allows unrestricted continuous actions. Panic bank runs arise as cascades of withdrawals by strategic complementarity and heuristics.

There are two distinct patterns of runs. For immediate runs, noticeable withdrawals take place right after the shock and follow stable increasing trajectories. On the contrary, for sudden runs, massive withdrawals burst out in a very short time window without visible signs. The paper is able to characterize how fast and how frequent bank runs occurs. Furthermore, we provide explicit computation of the critical point where the panic burst out, defined as tipping point. These results might be useful to devise interventions in time of crisis.

The model is simple and has several limitations. First, in time of crisis, individual decisions are likely to be correlated. To make the model more realistic, one can allow depositors to observe and learn from the actions of others. Second, the bank does not react to runs. One can introduce short-term borrowing with liquidity cost or fire sales of assets to have a richer set of dynamics. These limits serve as directions for future research.

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Appendix

Proof of Proposition 1

Solve the following differential equation by integration

$$\frac{dR}{dt} = \frac{1}{2\varepsilon}(\varepsilon - L^{\theta} + R(t))(1 - R(t))$$

Define $\alpha = \frac{1}{2\varepsilon}$ and $m = \varepsilon - L^{\theta}$, we obtain

$$\frac{dR}{(m + R(t))(1 - R(t))} = \alpha dt$$

The fraction on the LHS can be expressed as a sum:

$$\frac{1}{(m + R(t))(1 - R(t))} = \frac{A}{m + R(t)} + \frac{B}{1 - R(t)}$$

It is easy to show that $A = B = \frac{1}{m+1}$. Define $k = \frac{1}{m+1}$ and integrate both side. The LHS is given by

$$\int \frac{k}{m + R(t)} dR + \int \frac{k}{1 - R(t)} dR = k.ln(m + R(t)) - k.ln(1 - R(t)) + c_1 + c_2$$

where c_1, c_2 are integration constants. It is straightforward for the RHS:

$$\int \alpha dt = \alpha t + c_3$$

Taking exponential of both sides and rearranging terms yield

$$\left(\frac{m + R(t)}{1 - R(t)}\right)^k = e^{\alpha t}.e^{(c_3 - c_1 - c_2)}$$

To solve for $R(t)$, take both sides to the power of $\frac{1}{k}$, define $C = e^{\frac{c_3 - c_1 - c_2}{k}}$, we obtain the result after arraging terms:

$$R(t) = \frac{Ce^{ht} - m}{Ce^{ht} + 1}$$

with $h = \frac{\alpha}{k} = \alpha(m+1) = \frac{\varepsilon - L^{\theta} + 1}{2\varepsilon}$.
Plug the initial condition $R(0) = r_0$ into the equation yields: $C = \frac{r_0 + m}{1 - r_0}$. $\square$

Proof of Proposition 2

To find the exact time when liquidity reserve is depleted, we solve the following equation for t:

$$L = \frac{C.e^{ht} - m}{C.e^{ht} + 1}$$

with $m = \varepsilon - L^{\theta}, h = \frac{1}{2\varepsilon}(m+1), C = \frac{r_0 + m}{1 - r_0}$.
Arranging terms yields

$$e^{ht} = \frac{1}{C} \cdot \frac{m+L}{1-L}$$

Taking natural logarithm of both sides and arraging terms yield the final result

$$t^* = \frac{2\varepsilon}{\varepsilon - L^{\theta} + 1} \ln \left[\frac{\left(\varepsilon - L^{\theta} + L\right)(1 - r_0)}{\left(r_0 + \varepsilon - L^{\theta}\right)(1 - L)} \right]$$

□

Proof of Corollary 1-2

Let us remind that the condition $(r_0 > L^{\theta} - \varepsilon)$ holds.

If $r_0 > L$ then we have $\frac{\left(\varepsilon - L^{\theta} + L\right)(1-r_0)}{\left(r_0 + \varepsilon - L^{\theta}\right)(1-L)} < 1$, that is $t^* < 0$. It implies that $R(t^*) = L$ only holds for $r_0 < L$.

From the equation of t^*, it is trivial that t^* is strictly decreasing with respect to r_0.

Let denote $t^*(L) = \frac{2\varepsilon}{\varepsilon - L^{\theta} + 1} \ln \left[\frac{\left(\varepsilon - L^{\theta} + L\right)(1-r_0)}{\left(r_0 + \varepsilon - L^{\theta}\right)(1-L)} \right]$

We have

$$\begin{aligned} \frac{dt^*}{dL} &= \frac{2\varepsilon\theta L^{\theta-1}}{\varepsilon - L^{\theta} + 1} \ln \left[\frac{\left(\varepsilon - L^{\theta} + L\right)(1 - r_0)}{\left(r_0 + \varepsilon - L^{\theta}\right)(1 - L)} \right] \\ &\quad + \frac{2\varepsilon}{\varepsilon - L^{\theta} + 1} \left(\frac{1}{L + \varepsilon - L^{\theta}} + \frac{L}{1 - L} + \theta L^{\theta-1} \left(\frac{L - r_0}{\left(r_0 + \varepsilon - L^{\theta}\right)\left(L + \varepsilon - L^{\theta}\right)} \right) \right) \end{aligned}$$

None of these terms can be negative for any $L \in \left(r_0, (r_0 + \varepsilon)^{\frac{1}{\theta}}\right)$ with $0 < r_0 < 1$ and $\theta > 1$, therefore $\frac{dt^*}{dL} > 0$.

The dependency of t^* with respect to ε is more ambiguous. It is worth denoting t^* as $t^*(\varepsilon)$

$$\frac{dt^*}{d\varepsilon} = \frac{2\left(1 - L^{\theta}\right)}{\left(\varepsilon - L^{\theta} + 1\right)^2} \ln \left[\frac{\left(\varepsilon - L^{\theta} + L\right)(1 - r_0)}{\left(r_0 + \varepsilon - L^{\theta}\right)(1 - L)} \right] - \frac{2\varepsilon}{\varepsilon - L^{\theta} + 1} \frac{L - r_0}{\left(r_0 + \varepsilon - L^{\theta}\right)\left(L + \varepsilon - L^{\theta}\right)}$$

Devide both side by t^* yields

$$\left(\frac{1}{t^*}\right)\frac{dt^*}{d\varepsilon}=\frac{1}{\varepsilon\left(\varepsilon-L^\theta+1\right)}\left[\left(1-L^\theta\right)-\frac{2\varepsilon^2\left(L-r_0\right)}{\left(r_0+\varepsilon-L^\theta\right)\left(L+\varepsilon-L^\theta\right)}\left(\frac{1}{t^*}\right)\right]$$

The first term is always positive, the second term is negative if and only if

$$1-L^\theta<\frac{2\varepsilon^2\left(L-r_0\right)}{\left(r_0+\varepsilon-L^\theta\right)\left(L+\varepsilon-L^\theta\right)}\left(\frac{1}{t^*}\right)$$

Define

$$\tau\left(\varepsilon\right)=\frac{2\varepsilon^2\left(L-r_0\right)}{\left(r_0+\varepsilon-L^\theta\right)\left(L+\varepsilon-L^\theta\right)\left(1-L^\theta\right)}$$

If $t^*(\varepsilon)<\tau\left(\varepsilon\right)$, then $\frac{dt^*}{d\varepsilon}<0$. It is straighforward that for small ε, we have $t\left(\varepsilon\right)<\tau\left(\varepsilon\right)$, thus $t\left(\varepsilon\right)$ is a decreasing function of ε.

To obtain a general result, we need to consider

$$\tau'\left(\varepsilon\right)=2\frac{L-r_0}{1-L^\theta}\left[\left(-L^\theta+\frac{L+r_0}{2}\right)\varepsilon+\left(L-L^\theta\right)\left(r_0-L^\theta\right)\right]$$

If $L^\theta-\varepsilon<r_0<2L^\theta-L$ and $\ln\left(\frac{1-r_0}{1-L}\right)<\frac{L-r_0}{1-L^\theta}$, then $\tau\left(\varepsilon\right)$ is a decreasing function of ε and its graph is above the graph of $t\left(\varepsilon\right)$. Then $t\left(\varepsilon\right)$ is an increasing function of ε.

If $L^\theta-\varepsilon<r_0<L^\theta$ and $\ln\left(\frac{1-r_0}{1-L}\right)>\frac{L-r_0}{1-L^\theta}$, it implies that there exists a unique ε^* such that $t^*\left(\varepsilon\right)$ is a decreasing function of ε if $\varepsilon<\varepsilon^*$ and $t\left(\varepsilon\right)$ is an increasing function with ε if $\varepsilon>\varepsilon^*$.

If $2L^\theta-L<r_0<L^\theta$ and $\ln\left(\frac{1-r_0}{1-L}\right)<\frac{L-r_0}{1-L^\theta}$, then $\tau\left(\varepsilon\right)$ has a minimum value and crosses twice the graph of $t\left(\varepsilon\right)$. It means that for small values of ε, then $t\left(\varepsilon\right)$ is a decreasing function of ε, then for greater values it is an increasing function of ε, then a decreasing function of ε when ε increases.
□

Proof of Corollary 4-5

Compute the derivative of $(r_0+\varepsilon)^{\frac{1}{\theta}}$ with respect to r_0 yields:

$$\frac{\partial}{\partial r_0}(r_0+\varepsilon)^{\frac{1}{\theta}}=\frac{1}{\theta}(r_0+\varepsilon)^{\frac{1}{\theta}-1}$$

The bank run zone expands with r_0 if the derivative is greater than 1

$$\frac{1}{\theta}(r_0+\varepsilon)^{\frac{1}{\theta}-1}>1$$

$$r_0+\varepsilon<\theta^{\frac{\theta}{1-\theta}}$$

$$r_0<\theta^{\frac{\theta}{1-\theta}}-\varepsilon$$

The sign changed because $\theta>1$.

Next, we procced to prove that $\frac{\partial L^*}{\partial r_0} > 0$. Denote the function of hitting time in equation (5) as :

$$F(L^*, r_0) = t^*$$

Consider the relation:

$$\phi(L^*, r_0) = F(L^*, r_0) - t^*$$

The following conditions are satisfied:

1. $\phi(L^*, r_0) = F(L^*, r_0) - t^* = 0$
2. $\frac{\partial \phi(L^*, r_0)}{\partial L^*} = \frac{\partial F}{\partial L^*} > 0$

Thus, there exist an implicit function $L^*(r_0)$ such that:

$$F(L^*(r_0), r_0) = t^*$$

Using implicit differentiation with respect to r_0:

$$\frac{\partial F}{\partial r_0} + \frac{\partial F}{\partial L^*} . \frac{\partial L^*}{\partial r_0} = 0$$

$$\frac{\partial L^*}{\partial r_0} = -\frac{\frac{\partial F}{\partial r_0}}{\frac{\partial F}{\partial L^*}}$$

We have $\frac{\partial F}{\partial r_0} < 0$ and $\frac{\partial F}{\partial L^*} > 0$, therefore $\frac{\partial L^*}{\partial r_0} > 0$. The proof for ε is similar and therefore omitted. □

Proof of Proposition 5

Let $R'(t) = f(R)$, where $f(R) = \frac{1}{2\varepsilon}\left(\varepsilon - L^\theta + R\right)(1-R)$. When $R(t)$ is close to r_0, a Taylor expansion of f in the neighborhood of r_0 enables us to rewrite the dynamics as a constant growth rate model as follows:

$$f(R) = \frac{1}{2\varepsilon}(1-r_0)\left(\varepsilon - L^\theta + r_0\right) + (R - r_0)\frac{1}{2\varepsilon}(1 - 2r_0 - \varepsilon + L^\theta) + o\left((R-r_0)^2\right)$$

As long as the trajectory remains close to r_0, it satisfies

$$(R(t) - r_0)' = \frac{1}{2\varepsilon}(1-r_0)\left(\varepsilon - L^\theta + r_0\right) + (R - r_0)\frac{1}{2\varepsilon}(1 - 2r_0 - \varepsilon + L^\theta)$$

thus

$$R(t) \simeq r_0 - \frac{(1-r_0)\left(\varepsilon - L^\theta + r_0\right)}{(1 - 2r_0 - \varepsilon + L^\theta)} + \frac{(1-r_0)\left(\varepsilon - L^\theta + r_0\right)}{(1 - 2r_0 - \varepsilon + L^\theta)} e^{\frac{1}{2\varepsilon}(1-2r_0-\varepsilon+L^\theta)t}$$

To quantify t_I, we solve $R(t_I) - r_0 = \eta$. Thus t_I is the solution of

$$\eta = \left(e^{\frac{1}{2\varepsilon}(1-2r_0-\varepsilon+L^\theta)t_I} - 1\right)\frac{(1-r_0)\left(\varepsilon - L^\theta + r_0\right)}{(1 - 2r_0 - \varepsilon + L^\theta)}$$

that is, if $\frac{1-\varepsilon+L^{\theta}}{2} > r_0$

$$t_I = \frac{2\varepsilon}{(1-2r_0-\varepsilon+L^{\theta})} \ln\left(\frac{\eta(1-2r_0-\varepsilon+L^{\theta})}{(1-r_0)\left(\varepsilon-L^{\theta}+r_0\right)}+1\right)$$

With the condition $r_0 > L^{\theta}-\varepsilon$, a Taylor expansion in the neighborhood of η yields:

$$t_I = \frac{2\eta\varepsilon}{(1-r_0)\left(\varepsilon-L^{\theta}+r_0\right)}$$

□

Proof of Proposition 6

Let us denote the dynamics as $R'(t) = f(R)$, where $f(R) = a(1-R)(m+R)$, with $m = \varepsilon - L^{\theta}$ and $a = \frac{1}{2\varepsilon}$.

We need to characterize the tipping point where the curvature of the trajectory is maximum. The curvature of a curve $(t, R(t))$ is defined as

$$\kappa(t) = \frac{R''(t)}{\left(1+R'(t)^2\right)^{\frac{3}{2}}}$$

We can express the curvature as a function of R:

$$\widetilde{\kappa}(R) = \frac{f(R)\, f'(R)}{\left(1+f(R)^2\right)^{3/2}}$$

It is worth seeing that $\widetilde{\kappa}(R)$ is anti-symmetric according to $\frac{m-1}{2}$ (that is $\widetilde{\kappa}(R) = -\widetilde{\kappa}(1-m-R)$). We can therefore consider $\gamma(R) = \widetilde{\kappa}(R+\frac{1-m}{2})$, which is a translation of $\widetilde{\kappa}(R)$.

$$\gamma(R) = \frac{-2a^2(A+R)(A-R)R}{\left(1+a^2(A+R)(A-R)\right)^{3/2}}$$

where $A = \frac{1+m}{2}$.The optimum of $\gamma(R)$ on $[-A, A]$ is an interior point and is reached when the numerator $M(R)$ of its derivative is equal to 0 (we would check a posteriori conditions for which it is an interior point on $\left[r_0 - \frac{1-m}{2}, A\right]$).

As $M(R)$ is given by

$$M(R) = 3R^6 - 5\left(\frac{m+1}{2}\right)^2 R^2 + \left(\left(\frac{m+1}{2}\right)^4 - \frac{3}{a^2}\right)R + \left(\frac{m+1}{2}\right)^6 + (m+1)^2\varepsilon^2$$

solutions of $M(R) = 0$ are given by $R = \pm\sqrt{r}$, where r is a non negative solution of

$$N(r) = 3r^3 - 5\left(\frac{m+1}{2}\right)^2 r^2 + \left(\left(\frac{m+1}{2}\right)^4 - 12\varepsilon^2\right)r + \left(\frac{m+1}{2}\right)^6 + (m+1)^2\varepsilon^2$$

As $M(0) > 0$ and $M(A) = M(-A) = -8A^2\varepsilon^2 < 0$, $M(R) = 0$ has at least 2 roots in the intervall $[-A, A]$ and 1 root ouside this intervall as $\lim_{R\to\infty} M(R) = \infty$

As $N(0) > 0$ and $\lim_{r\to\infty} N(r) = -\infty$, there is a priori either 1 or 3 negative real roots. According to properties of $M(R)$, $N(r) = 0$ has 1 negative real root and 2 positive real roots, and we have to consider the smallest positive root $\widehat{r}$ to characterize the root of $M(R) = 0$ in the intervall $[-A, A]$.

Thus $\gamma(R)$ is maximum at $\widehat{R} = \frac{1-m}{2} - \sqrt{\widehat{r}}$. According to the properties of the roots, we know that $\frac{1-m}{2} - \sqrt{\widehat{r}} \geq -m$. We have now to find conditions such that $\frac{1-m}{2} - \sqrt{\widehat{r}} \geq r_0$, that is $\frac{1-m}{2} - r_0 \geq \sqrt{\widehat{r}}$.

As $\widehat{r}$ is a root of $N(r)$ such that $N(r) > 0$ for $[0, \widehat{r}[$ and $N(r) < 0$ for $r \in]\widehat{r}, A^2[$, then condition $\frac{1-m}{2} - r_0 \geq \sqrt{\widehat{r}}$ is satisfied if $N\left[\left(\frac{1-m}{2} - r_0\right)^2\right] \leq 0$. $\square$